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Asymptotic Behavior of the Eigenfrequency of a One-Dimensional Linear Thermoelastic System*

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In this paper, we are concerned with the asymptotic behavior of the eigenvalues arising from a one-dimensional linear thermoelastic system with the Dirichlet–Dirichlet boundary condition. It is shown that the eigenfrequency asymptotically falls on two branches: one branch is along the negative horizontal axis in the complex plane and the other branch is asymptotic to the vertical line $\operatorname{Re}\lambda = -\gamma^2/2k$. These results lead to the exponential stability of the system and also provide a proof for the numerical simulation results by Liu and Zheng (1993, *Quart. Appl. Math.*, **51**, 535–545).

1. INTRODUCTION

In general, the one-dimensional linear thermoelasticity problem for a homogeneous rod of uniform cross section with Dirichlet–Dirichlet boundary condition can be formulated as (see [1])

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \gamma v_x(x,t) = 0 & 0 < x < 1, t > 0, \\ v_t(x,t) + \gamma u_{xt}(x,t) - k v_{xx}(x,t) = 0 & 0 < x < 1, t > 0, \\ u(i,t) = v(i,t) = 0, & i = 0, 1, t \geq 0, \end{cases} \tag{1}$$

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where $\gamma > 0, k > 0$ are physical constants indicated in [1]. The following result is proved in [1, 5]:

THEOREM 1. *Let* $\mathsf{H} = H_0^1(0,1) \times L^2(0,1) \times L^2(0,1)$. *Then, for any initial condition* $(u_0(\cdot), u_1(\cdot), \nu_0(\cdot))^T \in \mathsf{H}$, *there exists a unique solution to Eq.* (1) *with*

$$(u(\cdot,t), u_t(\cdot,t), \nu(\cdot,t))^T = e^{\mathbb{A}t}(u_0(\cdot), u_1(\cdot), \nu_0(\cdot))^T \in C[0,\infty; \mathsf{H}),$$

where $e^{\mathbb{A}t}$ *is the* C_0*-semigroup of contractions on* H *generated by the operator* $\mathbb{A} : D(\mathbb{A}) \to \mathsf{H}$,

$$\mathbb{A} = \begin{pmatrix} 0 & 1 & 0 \\ D^2 & 0 & -\gamma D \\ 0 & -\gamma D & kD^2 \end{pmatrix}$$

with $D = \partial/\partial x$ *and* $D(\mathbb{A}) = H^2(0,1) \times H_0^1(0,1) \times H_0^1(0,1) \cap \mathsf{H}$. *Furthermore, the spectrum-determined-growth condition holds,*

$$\omega(\mathbb{A}) = s(\mathbb{A}),$$

provided that $s(\mathbb{A}) > -k\pi^2$, *where* $s(\mathbb{A})$ *and* $\omega(\mathbb{A})$ *are defined by*

$$s(\mathbb{A}) = \sup\{\mathrm{Re}\,\lambda | \lambda \in \sigma(\mathbb{A})\},$$
$$\omega(\mathbb{A}) = \inf\{\omega | \exists M > 1 \text{ such that } \|e^{\mathbb{A}t}\| \le Me^{\omega t} \text{ for all } t \ge 0\}.$$

It was also shown in [1] that the semigroup $e^{\mathbb{A}t}$ is uniformly exponentially stable. In [2], numerical simulations show that the eigenfrequency of system (1) approaches asymptotically to a vertical line that is parallel to the imaginary axis and contained in the open left half of the complex plane. In this paper, we give a theoretical proof for this numerical conjecture. The main difficulty in studying (1) lies in the fact that for Dirichlet−Neumann or Neumann−Dirichlet boundary conditions, the characteristic equations are cubic algebraic equations. So it is relatively easy to find all the eigenvalues and analyse their asymptotic behavior, as already performed in [3]. However, for the Dirichlet−Dirichlet boundary condition, the characteristic equations become transcendental and we must find another way to analyse the asymptotic behavior. Readers who are interested in earlier results of these problems can consult [4, 6]. Finally, let us point out that our proof for the numerical conjecture is quite elementary and so, unfortunately, a little bit lengthy.

The presentation of this paper is organized as follows: we shall derive a characteristic equation for the eigenvalues of system (1) in Section 2, and carry out an asymptotic analysis on the eigenvalues in Section 3. In Section 4, we shall prove the existence of eigenvalues and show that there are two branches of eigenvalues for system (1): one branch is along the negative horizontal axis and another branch approaches the vertical line $\mathrm{Re}\,\lambda = -\gamma^2/2k$. The results are consistent with those numerical results obtained in [3] for problems with Dirichlet–Neumann or Neumann–Dirichlet boundary conditions.

2. CHARACTERISTIC EQUATION

In this section, we shall derive a characteristic equation for the eigenvalues of (1). It is easy to see that $\mathbb{A}^{-1}$ is compact in H and hence $\sigma(\mathbb{A})$, the spectrum of operator, $\mathbb{A}$ consists of eigenvalues only. Thus $\lambda \in \sigma(\mathbb{A})$ if and only if there exists $(\phi, \psi) \neq 0$ such that

$$\begin{cases} \lambda^2\phi(x) - \phi''(x) + \gamma'(x) = 0, \\ \lambda\psi(x) + \lambda\gamma\phi'(x) - k\psi''(x) = 0, \\ \phi(i) = \psi(i) = 0, \qquad i = 0, 1. \end{cases} \tag{2}$$

To eliminate ψ, we differentiate the first equation of (2) and substitute ψ'' into the second equation of (2) to obtain

$$\begin{cases} \lambda^2\phi(x) + \phi''(x) + \gamma\psi'(x) = 0, \\ \gamma\lambda\psi(x) + \lambda(\lambda k + \gamma^2)\phi'(x) - k\phi'''(x) = 0, \\ \phi(i) = 0 = \lambda(\lambda k + \gamma^2)\phi'(i) - k\phi'''(i), \qquad i = 0, 1. \end{cases} \tag{3}$$

If we further differentiate the second equation of (3) and substitute ψ' into the first equation of (3), we get

$$\begin{cases} k\phi'''(x) - \lambda(k\lambda + \gamma^2 + 1)\phi''(x) + \lambda^3\phi(x) = 0, \\ \phi(i) = 0 = \lambda(\lambda k + \gamma^2)\phi'(i) - k\phi'''(i), \quad i = 0, 1, \end{cases} \tag{4}$$

with

$$\gamma\lambda\psi(x) = k\phi'''(x) - \lambda(\lambda k + \gamma^2)\phi'(x). \tag{5}$$

Thus, the eigenvalue problem (2) is equivalent to finding a pair of $(\lambda, \phi) \in \mathbb{C} \times H_0^1(0,1)$ such that $\phi \neq 0$ and Eq. (4) holds. Indeed, we can say more,

LEMMA 2. *The general solution of* (4) *is*

$$\phi(x) = c_1 e^{a_1 x} + c_2 e^{-a_1 x} + c_3 e^{a_2 x} + c_4 e^{-a_2 x}, \tag{6}$$

where $c_i, i = 1, 2, 3, 4,$ *are arbitrary constants and*

$$\begin{cases} a_1 = \sqrt{\dfrac{\lambda}{2k}\left[k\lambda + \gamma^2 + 1 + \sqrt{\left(k\lambda + \gamma^2 + 1\right)^2 - 4k\lambda}\right]}, \\ a_2 = \sqrt{\dfrac{\lambda}{2k}\left[k\lambda + \gamma^2 + 1 - \sqrt{\left(k\lambda + \gamma^2 + 1\right)^2 - 4k\lambda}\right]}. \end{cases} \tag{7}$$

Proof. To see this, we notice that the characteristic equation of (4) is

$$ka^4 - \lambda\left(k\lambda + \gamma^2 + 1\right)a^2 + \lambda^3 = 0 \tag{8}$$

and the roots of which satisfy

$$a^2 = \frac{\lambda}{2k}\left[k\lambda + \gamma^2 + 1 \pm \sqrt{\left(k\lambda + \gamma^2 + 1\right)^2 - 4k\lambda}\right]. \tag{9}$$

So, the algebraic equation (8) will have four roots $a_1, -a_1, a_2, -a_2$, as listed in (7). ∎

We next look for a condition that the eigenvalue λ will satisfy. When $(k\lambda + \gamma^2 + 1)^2 - 4k\lambda = 0$, we have

$$\lambda = \frac{1 - \gamma^2}{k} \pm \frac{2\sqrt{\gamma}}{k} i. \tag{10}$$

For the case that $(k\lambda + \gamma^2 + 1)^2 - 4k\lambda \neq 0$, we can put the boundary conditions at $x = 0$ into (6), and obtain

$$-2g_1\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} (g_1 + g_2)c_3 + (g_1 - g_2)c_4 \\ (g_1 - g_2)c_3 + (g_1 + g_2)c_4 \end{pmatrix}, \tag{11}$$

where

$$g_i = a_i\left(k\lambda^2 + \gamma^2\lambda - ka_i^2\right) \neq 0, \qquad i = 1, 2. \tag{12}$$

Similarly, the boundary conditions at $x = 1$, yield

$$-2g_1\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \exp(a_2 - a_1)(g_1 + g_2)c_3 + \exp(-a_2 - a_1)(g_1 - g_2)c_4 \\ \exp(a_2 + a_1)(g_1 - g_2)c_3 + \exp(a_1 - a_2)(g_1 + g_2)c_4 \end{pmatrix}. \tag{13}$$

Thus, from (11) and (13), the necessary and sufficient condition for (4) to possess a nonzero solution ϕ is

$$\det\begin{pmatrix} [1 - \exp(a_2 - a_1)](g_1 + g_2) & [1 - \exp(-a_2 - a_1)](g_1 - g_2) \\ [1 - \exp(a_2 + a_1)](g_1 - g_2) & [1 - \exp(a_1 - a_2)](g_1 + g_2) \end{pmatrix} = 0, \tag{14}$$

which is just

$$8g_1g_2 - [\exp(a_1 - a_2) + \exp(a_2 - a_1)](g_1 + g_2)^2 + [\exp(-a_1 - a_2) + \exp(a_2 + a_1)](g_1 - g_2)^2 = 0. \tag{15}$$

After some simplifications (see the proof of the lemma below), we can further conclude that:

LEMMA 3. *The characteristic equation for the eigenvalue λ of problem* (2) *is*

$$\begin{aligned} &8\gamma^2\sqrt{k\lambda}\exp(a_1 + a_2) + [\exp(2a_1) + \exp(2a_2)]\left(k\lambda + \gamma^2 + 1 + 2\sqrt{k\lambda}\right) \\ &\quad\times\left(1 - \sqrt{k\lambda}\right)^2 \\ &-[1 + \exp(2a_1 + 2a_2)]\left(k\lambda + \gamma^2 + 1 - 2\sqrt{k\lambda}\right) \\ &\quad\times\left(1 + \sqrt{k\lambda}\right)^2 = 0. \end{aligned} \tag{16}$$

Proof. To see this, we just need to deduce from (12) and (9) that

$$g_1g_2 = a_1a_2\left[\lambda^2(k\lambda + \gamma^2)^2 - \lambda k(k\lambda + \gamma^2)(a_1^2 + a_2^2) + ka_1^2a_2^2\right],$$

$$g_1 - g_2 = (a_1 - a_2)\left(-\lambda - \lambda\sqrt{k\lambda}\right),$$

$$g_1 + g_2 = (a_1 + a_2)\left(-\lambda + \lambda\sqrt{k\lambda}\right),$$

$$a_1a_2 = \lambda\sqrt{\lambda}/\sqrt{k}, \qquad a_1^2 + a_2^2 = \lambda(k\lambda + \gamma^2 + 1)/k,$$

and also that

$$g_1 g_2 = -\frac{\gamma^2}{\sqrt{k}}\sqrt{\lambda}\cdot\lambda^3,$$

$$(g_1 - g_2)^2 = \frac{\lambda^3}{k}\left(k\lambda + \gamma^2 + 1 - 2\sqrt{k\lambda}\right)\left(1 + \sqrt{k\lambda}\right)^2,$$

$$(g_1 + g_2)^2 = \frac{\lambda^3}{k}\left(k\lambda + \gamma^2 + 1 + 2\sqrt{k\lambda}\right)\left(1 - \sqrt{k\lambda}\right)^2. \quad \blacksquare$$

Consequently, we can conclude that

PROPOSITION 4. *$\lambda \in \sigma(\mathbb{A})$ if and only if λ is given by* (10) *or λ is a root of Eq.* (16).

3. ASYMPTOTIC BEHAVIOR

In this section, we will give an asymptotic analysis for the roots of Eq. (16). To simplify our presentation, we will use λ instead of $k\lambda$ throughout our analysis, but return to $k\lambda$ when we draw our final conclusions. Thus, (16) is written as

$$\begin{aligned} &8\gamma^2\sqrt{\lambda}\exp(b_1 + b_2) \\ &\quad + \left[\exp(2b_1) + \exp(2b_2)\right]\left(\lambda + \gamma^2 + 1 + 2\sqrt{\lambda}\right)\left(1 - \sqrt{\lambda}\right)^2 \\ &\quad - \left[1 + \exp(2b_1 + 2b_2)\right]\left(\lambda + \gamma^2 + 1 - 2\sqrt{\lambda}\right)\left(1 + \sqrt{\lambda}\right)^2 = 0, \end{aligned} \tag{17}$$

where

$$b_1 = \frac{\lambda}{\sqrt{2}k} f(1/\lambda), \qquad b_2 = \frac{\sqrt{2\lambda}}{k}\frac{1}{f(1/\lambda)}, \tag{18}$$

and

$$f(1/\lambda) = \left[1 + \frac{\gamma^2 + 1}{\lambda} + \sqrt{\left(1 + \frac{\gamma^2 + 1}{\lambda}\right)^2 - \frac{4}{\lambda}}\,\right]^{1/2}.$$

The Taylor expansion for the complex functions $f(1/\lambda)$ and $1/f(1/\lambda)$ at $\lambda = \infty$ are

$$\begin{cases} f(1/\lambda) = \sqrt{2} + \dfrac{\gamma^2}{\sqrt{2}}\dfrac{1}{\lambda} + \dfrac{4\gamma^2 - \gamma^4}{2\sqrt{2}}\dfrac{1}{\lambda^2} + O(|\lambda|^{-3}), \\ 1/f(1/\lambda) = \dfrac{1}{\sqrt{2}} - \dfrac{\gamma^2}{2\sqrt{2}}\dfrac{1}{\lambda} + \dfrac{\gamma^4 - 2\gamma^2}{2\sqrt{2}}\dfrac{1}{\gamma^2} + O(|\lambda|^{-3}) \end{cases}. \tag{19}$$

And hence as $|\lambda| \to \infty$,

$$\begin{cases} b_1 = \dfrac{\lambda}{k} + \dfrac{\gamma^2}{2k} + O(|\lambda|^{-1}), \\ b_2 = \dfrac{\sqrt{\lambda}}{k} - \dfrac{\gamma^2}{2k}\dfrac{1}{\sqrt{\lambda}} + \dfrac{\gamma^4 - 2\gamma^2}{2k}\dfrac{1}{\lambda\sqrt{\lambda}} + O(|\lambda|^{-5/2}). \end{cases} \tag{20}$$

Now write

$$\lambda = |\lambda| e^{i\theta}.$$

As the closed right half plane does not contain any eigenvalues (because we always have $\mathrm{Re}\lambda < 0$ from the dissipativity of $\mathbb{A}$) and since λ is symmetric with respect to the real axis, we need only consider the case $\pi/2 < \theta \leq \pi$. We further divide our analysis into two parts and formulate the results into two lemmas, and then draw our concluding theorem from them. The two lemmas are as follows.

LEMMA 5. *Let $\delta > 0$ be sufficiently small. Then for all $\pi/2 + \delta < \theta \leq \pi$,*

$$\lambda = -(kn\pi)^2 + \gamma^2 + O(|\lambda|^{-1}),$$

where n is some positive integer.

LEMMA 6. *Let $\delta > 0$ be sufficiently small. Then for all $\pi/2 < \theta \leq \pi/2 + \delta$,*

$$\mathrm{Re}\lambda = -\frac{\gamma^2}{2} + O(|\lambda|^{-1}) \qquad \text{and} \qquad |\lambda| = kn\pi + O(|\lambda|^{-1}),$$

where n is some positive integer.

The proofs of the above two lemmas are very similar to each other. They both begin with extracting asymptotic estimates for b_2 and b_1 from (20) and applying them to (17) (a modification of (17) to be exact, please see (21) in the proofs) to yield estimates for $\exp(-2b_2)$ and $\exp(2b_1)$. Then,

$\exp(-2b_2)$ and respectively $\exp(2b_1)$ are split into two parts, $\exp(\mathrm{Re}(-2b_2))$ and $\exp(\mathrm{Im}(-2b_2))$ (respectively $\exp(\mathrm{Re}(2b_1))$ and $\exp(\mathrm{Im}(2b_1))$, to yield estimates for $\mathrm{Re}(-2b_2)$ (and respectively $\mathrm{Re}(2b_1)$). We then compare these estimates with those from (20) and draw the conclusions. The detailed proofs are as follows.

Proof of Lemma 5. We shall carry out an asymptotic analysis on (17) and draw our conclusion. To begin, we have

$$\begin{aligned}
\exp(b_1) &= \exp\left(\frac{\lambda}{k} + \frac{\gamma^2}{2k}\right)\exp\big(O(|\lambda|^{-1})\big)\\
&= \exp\left(\frac{|\lambda|\cos\theta}{k} + \frac{\gamma^2}{2k}\right)\exp\left(\frac{i|\lambda|\cos\theta}{k}\right)\exp\big(O(|\lambda|^{-1})\big)\\
&= O\big(\exp(-\gamma_1|\lambda|)\big), \qquad \text{for some } \gamma_1 > 0,\\
\exp(-b_2) &= \exp\left(-\frac{\sqrt{\lambda}}{k}\right)\exp\big(O(|\lambda|^{-1/2})\big)\\
&= \exp\left(-\frac{|\lambda|^{1/2}}{k}\cos\frac{\theta}{2}\right)\exp\big(O(|\lambda|^{-1/2})\big)\\
&= O(1).
\end{aligned}$$

Multiplying $\exp(-2b_2)$ on both sides of (17) yields

$$\begin{aligned}
&8\gamma^2\sqrt{\lambda}\exp(b_1 - b_2)\\
&\quad + [1 + \exp(2b_1 - 2b_2)]\big(\lambda + \gamma^2 + 1 + 2\sqrt{\lambda}\big)\big(1 - \sqrt{\lambda}\big)^2\\
&\quad - [\exp(2b_1) + \exp(-2b_2)]\big(\lambda + \gamma^2 + 1 - 2\sqrt{\lambda}\big)\big(1 + \sqrt{\lambda}\big)^2 = 0.
\end{aligned} \tag{21}$$

Thus

$$\begin{aligned}
&\big(\lambda + \gamma^2 + 1 + 2\sqrt{\lambda}\big)\big(1 - \sqrt{\lambda}\big)^2\\
&\quad - \exp(-2b_2)\big(\lambda + \gamma^2 + 1 - 2\sqrt{\lambda}\big)\big(1 + \sqrt{\lambda}\big)^2\\
&\quad = O\big(|\lambda|^2\exp(-\gamma_1|\lambda|)\big).
\end{aligned}$$

A simple calculation yields

$$\exp(-2b_2) = \frac{\left(\lambda + \gamma^2 + 1 + 2\sqrt{\lambda}\right)\left(1 - \sqrt{\lambda}\right)^2}{\left(\lambda + \gamma^2 + 1 - 2\sqrt{\lambda}\right)\left(1 + \sqrt{\lambda}\right)^2} + O\big(\exp(-\gamma_1|\lambda|)\big)$$
$$= 1 - 4\gamma^2\lambda^{-3/2} + O\big(|\lambda|^{-5/2}\big), \tag{22}$$

which will be our main building block to obtain our claimed assertion. Rewrite (22) as

$$e^{\mathrm{Re}(-2b_2)} = \left[1 - 4\gamma^2|\lambda|^{-3/2}\exp\left(-\frac{i3\theta}{2}\right)\right]e^{i\mathrm{Im}(2b_2)} + O\big(|\lambda|^{-5/2}\big)$$
$$= e^{i\mathrm{Im}(2b_2)} + O\big(|\lambda|^{-3/2}\big). \tag{23}$$

Comparing (23) with (22), we see that

$$1 - O\big(|\lambda|^{-3/2}\big) \le e^{i\mathrm{Im}(2b_2)} + O\big(|\lambda|^{-3/2}\big) \le 1 + O\big(|\lambda|^{-3/2}\big).$$

So

$$\begin{cases} e^{i\mathrm{Im}(2b_2)} = 1 + O\big(|\lambda|^{-3.2}\big), \\ e^{\mathrm{Re}(-2b_2)} = 1 + O\big(|\lambda|^{-3/2}\big). \end{cases}$$

On the other hand, we observe from (20) that

$$\mathrm{Re}(-2b_2) = -\frac{2|\lambda|^{1/2}}{k}\cos\frac{\theta}{2} + O\big(|\lambda|^{-1/2}\big),$$

and conclude immediately that

$$\theta \to \pi, \qquad \text{as } |\lambda| \to \infty, \tag{24}$$

because the left hand side above would not decrease to zero if otherwise. Indeed, we may even show that $\theta = \pi$ for large enough $|\lambda|$. To see this, we analyse (23) again. Since $e^{i\mathrm{Im}(2b_2)} = 1 + O(|\lambda|^{-3/2})$ so

$$\sin\big(\mathrm{Im}(2b_2)\big) = O\big(|\lambda|^{-3/2}\big)$$

and

$$\cos\big(\mathrm{Im}(2b_2)\big) = \sqrt{1 - \sin^2\big(\mathrm{Im}(2b_2)\big)} = \sqrt{1 + O\big(|\lambda|^{-3}\big)}$$
$$= 1 + O\big(|\lambda|^{-3}\big).$$

Applying them on (23),

$$\begin{aligned} e^{\mathrm{Re}(-2b_2)} &= \left[1 - 4\gamma^2|\lambda|^{-3/2}\exp\left(\frac{i3\theta}{2}\right)\right]e^{i\mathrm{Im}(2b_2)} + O\left(|\lambda|^{-5/2}\right) \\ &= \left[1 - 4\gamma^2|\lambda|^{-3/2}\exp\left(\frac{i3\theta}{2}\right)\right]\left[\cos\left(\mathrm{Im}(2b_2)\right) + i\sin\left(\mathrm{Im}(2b_2)\right)\right] \\ &\quad + \mathrm{O}\left(|\lambda|^{-5/2}\right) \\ &= 1 - 4\gamma^2|\lambda|^{-3/2}\exp\left(\frac{i3\theta}{2}\right) + i\sin\left(\mathrm{Im}(2b_2)\right) + O\left(|\lambda|^{-5/2}\right), \end{aligned}$$

which implies, via comparing the real and imaginary parts, that

$$\mathrm{Re}(-2b_2) = -4\gamma^2|\lambda|^{-3/2}\cos\frac{3\theta}{2} + O\left(|\lambda|^{-5/2}\right).$$

Putting this into (20), we obtain

$$\begin{aligned} &\left[-\frac{2|\lambda|^{1/2}}{k} + \frac{\gamma^2}{k|\lambda|^{1/2}}\right]\cos\frac{\theta}{2} - \frac{\gamma^4 - 2\gamma^2}{k|\lambda|^{3/2}}\cos\frac{3\theta}{2} \\ &\qquad = -4\gamma^2|\lambda|^{-3/2}\cos\frac{3\theta}{2} + O\left(|\lambda|^{-5/2}\right), \end{aligned} \tag{25}$$

which implies that $\cos(\theta/2) = O(|\lambda|^{-2})$. If $\theta \neq \pi$ for large $|\lambda|$, then

$$\cos\frac{3\theta}{2}\Big/\cos\frac{\theta}{2} \to -3,$$

and (25) leads to a contradiction that $|\lambda|^{1/2} = O(|\lambda|^{-1/2})$. Thus

$$\theta = \pi, \qquad \text{for large } |\lambda|. \tag{26}$$

Finally, from (20) and (26),

$$\begin{aligned} \mathrm{Im}(2b_2) &= \left[\frac{2|\lambda|^{1/2}}{k} + \frac{\gamma^2}{k|\lambda|^{1/2}}\right]\sin\frac{\theta}{2} + O\left(|\lambda|^{-3/2}\right) \\ &= \frac{2|\lambda|^{1/2}}{k} + \frac{\gamma^2}{k|\lambda|^{1/2}} + O\left(|\lambda|^{-3/2}\right). \end{aligned}$$

Since $\sin(\operatorname{Im}(2b_2)) = O(|\lambda|^{-3/2})$, we have

$$\frac{2|\lambda|^{1/2}}{k} + \frac{\gamma^2}{k|\lambda|^{1/2}} = 2n\pi + O\left(|\lambda|^{-3/2}\right)$$

or

$$\lambda = -(kn\pi)^2 + \gamma^2 + O\left(|\lambda|^{-1}\right), \tag{27}$$

where n is a sufficiently large integer. ∎

Proof of Lemma 6. In this case, $\pi/4 < \theta/2 \le \pi/4 + \delta/2$. Thus, there is some $\gamma_2 > 0$ such that

$$\begin{aligned}\exp(-b_2) &= \exp\left(-\frac{\sqrt{\lambda}}{k}\right)\exp\left(O\left(|\lambda|^{-1/2}\right)\right)\\ &= \exp\left(-\frac{|\lambda|^{1/2}}{k}\cos\frac{\theta}{2}\right)\exp\left(O\left(|\lambda|^{-1/2}\right)\right)O\left(\exp\left(-\gamma_2|\lambda|^{1/2}\right)\right)\end{aligned}$$

and

$$\begin{aligned}\exp(b_1) &= \exp\left(\frac{\lambda}{k} + \frac{\gamma^2}{2k}\right)\exp\left(O\left(|\lambda|^{-1}\right)\right)\\ &= \exp\left(\frac{|\lambda|}{k}\cos\theta + \frac{\gamma^2}{2k}\right)\exp\left(\frac{i|\lambda|}{k}\sin\theta\right)\exp\left(O\left(|\lambda|^{-1}\right)\right) = O(1).\end{aligned}$$

The rest of the proof is very similar to that of Lemma 5. We apply these two estimates on (21) and deduce

$$\begin{aligned}&\left(\lambda + \gamma^2 + 1 + 2\sqrt{\lambda}\right)\left(1 - \sqrt{\lambda}\right)^2 - \exp(2b_1)\left(\lambda + \gamma^2 + 1 - 2\sqrt{\lambda}\right)\left(1 + \sqrt{\lambda}\right)^2\\ &\qquad = O\left(|\lambda|^2\exp\left(-\gamma_2|\lambda|^{1/2}\right)\right)\end{aligned}$$

and

$$\begin{aligned}\exp(2b_1) &= \frac{\left(\lambda + \gamma^2 + 1 + 2\sqrt{\lambda}\right)\left(1 - \sqrt{\lambda}\right)^2}{\left(\lambda + \gamma^2 + 1 - 2\sqrt{\lambda}\right)\left(1 + \sqrt{\lambda}\right)^2} + O\left(\exp\left(-\gamma_2|\lambda|^{1/2}\right)\right)\\ &= 1 + O\left(|\lambda|^{-3/2}\right).\end{aligned} \tag{28}$$

Using the exact argument as in Lemma 5, we write (28) as

$$e^{\operatorname{Re}(2b_1)} = e^{i\operatorname{Im}(-2b_1)} + O\left(|\lambda|^{-3/2}\right),$$

and conclude, via comparing it with (28), that

$$\begin{cases} e^{\mathrm{Re}(2b_1)} = 1 + O(|\lambda|^{-3/2}), \\ e^{i\mathrm{Im}(-2b_1)} = 1 + O(|\lambda|^{-3/2}). \end{cases} \tag{29}$$

Since (20) implies that

$$\begin{cases} \mathrm{Re}(2b_1) = \dfrac{2}{k}|\lambda|\cos\theta + \dfrac{\gamma^2}{k} + O(|\lambda|^{-1}), \\ \mathrm{Im}(2b_1) = \dfrac{2}{k}|\lambda|\sin\theta + O(|\lambda|^{-1}), \end{cases} \tag{30}$$

we immediately conclude that

$$\theta \to \pi/2, \qquad \text{as } |\lambda| \to \infty. \tag{31}$$

Furthermore, by following those arguments in the proof of Lemma 4, we can compare (20) with (29) to yield

$$\mathrm{Re}(2b_1) = O(|\lambda|^{-3/2}).$$

Putting this into (30), we have

$$\frac{2}{k}|\lambda|\cos\theta + \frac{\gamma^2}{k} + O(|\lambda|^{-1}) = O(|\lambda|^{-3/2})$$

and hence,

$$\frac{2}{k}|\lambda|\cos\theta + \frac{\gamma^2}{k} = O(|\lambda|^{-1}),$$

which is

$$\cos\theta = -\frac{\gamma^2}{2|\lambda|} + O(|\lambda|^{-2}). \tag{32}$$

Since $\cos\theta = -\mathrm{Re}\lambda/|\lambda|$, (32) then implies

$$\mathrm{Re}\lambda = -\frac{\gamma^2}{2} + O(|\lambda|^{-1}). \tag{33}$$

Finally, from (32) and the second estimation of (30), we see that

$$\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{1 + O(|\lambda|^{-2})} = 1 + O(|\lambda|^{-2}),$$

$$\mathrm{Im}(-2b_1) = \frac{2}{k}|\lambda|\sin\theta + O(|\lambda|^{-1}) = \frac{2}{k}|\lambda| + O(|\lambda|^{-1}).$$

These together with (29) yield

$$|\lambda| = kn\pi + O\left(|\lambda|^{-1}\right), \tag{34}$$

where n is a sufficiently large integer. ∎

Combining Lemmas 5 and 6, we have proved that

THEOREM 7. *Asymptotically, there are at most two branches for the eigenvalues of operator $\mathbb{A}$. One branch is along the negative axis and the other branch approaches a vertical line* $\operatorname{Re}\lambda = -\gamma^2/2k$ *parallel to the imaginary axis.*

4. THE EXISTENCE OF THE EIGENVALUES

In this section, we shall establish the existence of the eigenvalues of the operator $\mathbb{A}$. Our main tool is Rouché's Theorem. As in Section 3, we separate our discussion into two cases.

Case I. First, we assume that $\lambda = |\lambda|e^{i\theta}$, $\pi/2 + \delta < \theta \le \pi$, with $\delta > 0$ sufficiently small. Then by (22),

$$\exp(-2b_2) = 1 + O\left(|\lambda|^{-3/2}\right). \tag{35}$$

From (20), $-2b_2 = -2(\sqrt{\lambda}/k) + \gamma^2/(k\sqrt{\lambda}) + O(|\lambda|^{-3/2})$, so (35) implies that

$$\exp\left(-2\frac{\sqrt{\lambda}}{k} + \frac{\gamma^2}{k}\frac{1}{\sqrt{\lambda}}\right) = 1 + O\left(|\lambda|^{-3/2}\right). \tag{36}$$

It is easy to see that $\exp(-2(\sqrt{\lambda}/k) + \gamma^2/(k\sqrt{\lambda}) = 1$ has solutions

$$\lambda_n = -(kn\pi)^2 + \gamma^2 + O\left(|\lambda_n|^{-1}\right). \tag{37}$$

Let $\sqrt{\lambda}$ be the positive branch of the square root of complex λ, and so the mapping $\lambda \to \sqrt{\lambda}$ maps the negative real axis of the complex plane to the positive imaginary axis. Instead of (36), we let $\mu := = \sqrt{\lambda}$ and consider

$$\exp\left(-2\frac{\mu}{k} + \frac{\gamma^2}{k}\frac{1}{\mu}\right) = 1 + O\left(|\mu|^{-3}\right). \tag{38}$$

Let

$$d = -2\frac{\mu}{k} + \frac{\gamma^2}{k}\frac{1}{\mu}. \tag{39}$$

Let O_n be a circle with radius $\alpha\lambda_n^{-1/2}$, $\alpha > 0$, and centered at $\sqrt{-\lambda_n} = i\sqrt{\lambda_n}$. That is,

$$O_n := \left\{ \mu \in \mathbb{C} : \mu = i\sqrt{\lambda_n} + \alpha\lambda_n^{-1/2} e^{i\theta}, 0 \le \theta \le 2\pi \right\}.$$

For all $\mu \in O_n$, we have

$$|\mu| = \sqrt{\lambda_n}\left[1 + O\left(\lambda_n^{-1}\right)\right]. \tag{40}$$

So, from (39) and (37), we conclude for all $\mu \in O_n$ that

$$\begin{aligned}
\operatorname{Re} d &= -\frac{2\alpha}{k}\lambda_n^{-1/2}\cos\theta + O\left(\lambda_n^{-3/2}\right), \\
\operatorname{Im} d &= -\frac{2}{k}\sqrt{\lambda_n} - \frac{2\alpha}{k}\lambda_n^{-1/2}\sin\theta - \frac{\gamma^2}{k}|\mu|^{-2}\sqrt{\lambda_n} + O\left(\lambda_n^{-3/2}\right), \\
&= -\frac{2}{k}\sqrt{\lambda_n} - \frac{2\alpha}{k}\lambda_n^{-1/2}\sin\theta - \frac{\gamma^2}{k}\lambda_n^{-1/2} + O\left(\lambda_n^{-3/2}\right), \\
&= -2n\pi - \frac{2\alpha}{k}\lambda_n^{-1/2}\sin\theta + O\left(\lambda_n^{-3/2}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
|1 - \exp(d)|^2 &= \left[1 - \exp(\operatorname{Re} d)\right]^2 + 2\exp(\operatorname{Re} d)\left[1 - \cos(\operatorname{Im} d)\right] \\
&= \frac{4\alpha^2}{k^2}\lambda_n^{-1} + O\left(\lambda_n^{-3/2}\right).
\end{aligned} \tag{41}$$

Since $i\sqrt{\lambda_n}$ is the unique root of $\exp(d) = 1$ inside O_n, we can apply Rouché's theorem to functions $\exp(d) - 1$ and $\exp(d) - 1 - O(|\mu|^{-3})$ and conclude that there exists a unique zero μ_n for the equation $\exp(d_2) - 1 - O(|\mu|^{-3}) = 0$ inside O_n for all $n \ge N$ when N is a sufficiently large integer.

Let $\hat{\lambda}_n = \mu_n^2$. Since μ_n is inside O_n, we know that

$$|\hat{\lambda}_n + \lambda_n| \le 2\alpha + O\left(\lambda_n^{-1}\right), \qquad \text{for all } n \ge N. \tag{42}$$

Thus, $\hat{\lambda}_n$ is the unique root of (36), and hence of (35), in the order of $O(n^2)$. Furthermore, by the arbitrariness of $\alpha > 0$, we see that

$$\lim_{n \to \infty} |\hat{\lambda}_n + \lambda_n| = 0. \tag{43}$$

Case II. As for now, we assume that $\lambda = |\lambda| e^{i\theta}, \pi/2 < \theta \leq \pi/2 + \delta$, with $\delta > 0$ sufficiently small. From (28)

$$\exp(2b_1) = 1 + O\big(|\lambda|^{-3/2}\big). \tag{44}$$

Since $b_1 = \lambda/k + \gamma^2/(2k) + O(|\lambda|^{-1})$, so (44) implies that

$$\exp\left(2\frac{\lambda}{k} + \frac{\gamma^2}{k}\right) = 1 + O\big(|\lambda|^{-1}\big), \tag{45}$$

and we notice that $\exp(2\lambda/k + \gamma^2/k) = 1$ possesses the solutions

$$\sigma_n = -\frac{\gamma^2}{2} + in\pi. \tag{46}$$

Let $\hat{O}_n$ be the circle with radius $|\sigma_n|^{-1/2}$ and centered at σ_n:

$$\hat{O}_n := \left\{\lambda \in \mathbb{C} : \lambda = \sigma_n + |\sigma_n|^{-1/2} e^{i\theta}, 0 \leq \theta \leq 2\pi\right\}.$$

Then for all $\lambda \in \hat{O}_n$ we have

$$|\lambda| = |\sigma_n|\Big[1 + O\big(|\sigma_n|^{-3/2}\big)\Big].$$

Let $c = 2\lambda/k + \gamma^2/k$, then for all $\lambda \in \hat{O}_n$,

$$\operatorname{Re} c = \frac{2}{k}|\sigma_n|^{-1/2}\cos\theta$$

$$\operatorname{Im} c = 2n\pi + \frac{2}{k}|\sigma_n|^{-1/2}\sin\theta$$

and so

$$\begin{aligned}|1 - \exp(c)|^2 &= [1 - \exp(\operatorname{Re} c)]^2 + 2\exp(\operatorname{Re} c)[1 - \cos(\operatorname{Im} c)]\\ &= \frac{4}{k^2}|\sigma_n|^{-1} + O\big(|\sigma_n|^{-3/2}\big).\end{aligned}$$

Applying Rouché's theorem to functions $\exp(2\lambda/k + \gamma^2/k) - 1$ and $\exp(2\lambda/k + \gamma^2/k) - 1 - O(|\lambda|^{-1})$, we conclude that there exists a unique zero $\hat{\sigma}_n$ for $\exp(2\lambda/k + \gamma^2/k) - 1 - O(|\lambda|^{-1}) = 0$ inside $\hat{O}_n$ for all $n \geq \hat{N}$ when $\hat{N}$ is a sufficiently large integer.

Since $\hat{\sigma}_n$ lies inside $\hat{O}_n$, by definition,

$$|\hat{\sigma}_n - \sigma_n| \leq |\sigma_n|^{-1/2}, \qquad \text{for all } n \geq \hat{N}. \tag{47}$$

This together with (34) implies that this $\hat{\sigma}_n$ is the unique root of (44) in the order of $O(n^{-1})$.

Combining these results with Theorem 7, we can conclude that

THEOREM 8. *The eigenvalues of the operator* $\mathbb{A}$ *consist of a real sequence* $\{\hat{\lambda}_n/k\}_{n=1}^{\infty}$ *and a sequence of conjugate pairs* $\{\hat{\sigma}_n/k, \overline{\hat{\sigma}}_n/k\}_{n=1}^{\infty}$ *with*

$$\begin{cases} \hat{\lambda}_n/k = -k(n\pi)^2 + \gamma^2/k + O(n^{-2}), \\ \hat{\sigma}_n/k = -\gamma^2/(2k) + in\pi + O(n^{-1}). \end{cases} \tag{48}$$

Applying these results on Theorem 1, we have the following stability results.

COROLLARY 9. *Let* $s(\mathbb{A})$ *be the spectral bound of operator* $\mathbb{A}$ *defined in Theorem* 1 *and let* $\omega = -s(\mathbb{A})$. *Then* $0 < \omega < \gamma^2/(2k)$, *and for any* $\varepsilon > 0$ *sufficiently small, there exists a positive number* $M > 1$ *such that*

$$E(t) \le Me^{(-\omega+\varepsilon)t}E(0), \tag{49}$$

where

$$E(t) = \frac{1}{2}\int_0^1 \left[u_t^2(x,t) + u_x^2(x,t) + v^2(x,t)\right] dx \tag{50}$$

is the energy function of system (1).

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